ON THE COMPLETE SYSTEM OF EQUATIONS OF A COMPRESSIBLE VISCO-PLASTIC FLUID

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PMM Vol.23, No.6, 1959, pp. 1142-1143

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(Received 29 August 1959)

It is essential, in principle at least, to allow for compressibility, when obtaining the solution to a number of problems connected with visco-plastic flows. Amongst such problems are those dealing with the propagation of both acoustic and large disturbances through a visco-plastic fluid. However, there does not appear to be any complete system of equations in the literature which describes the motion of a compressible visco-plastic fluid.

We are proposing such a system in the present note.

The complete system of equations describing any arbitrary motion of an incompressible visco-plastic fluid is contained in reference [1].

The relations between the stress tensor components and velocities of deformation are as follows:

$$p_{xx} = -p + 2\left(\mu + \frac{\tau_s}{h}\right)\left(\frac{\partial u}{\partial x} - \frac{1}{3}\operatorname{div}\bar{v}\right), \qquad p_{xy} = \left(\mu + \frac{\tau_s}{h}\right)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \text{ etc.} \tag{1}$$

The formulas which have not been written down here are obtained by cyclic permutation of the coordinates, μ is the coefficient of viscosity, τ_s is the constant of plasticity,

$$h = \sqrt{\frac{2}{3}}\sqrt{(s_{xx} - s_{yy})^2 + (s_{yy} - s_{zz})^2 + (s_{zz} - s_{xx})^2 + \frac{3}{2}(s_{xy}{}^2 + s_{yz}{}^2 + s_{zx}{}^2)}$$

$$s_{xx} = \frac{\partial u}{\partial x}, \ldots, \qquad s_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \ldots$$

p is the pressure, $V(u, v, w)$ the velocity vector. The expressions (1) which we will consider applicable to a compressible fluid together with the equations of motion of an arbitrary continuous medium and the equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho V) = 0 \tag{2}$$

do not constitute a complete system. In order to close the system we must assume that there exists a relation between pressure p and density ρ only (equation of state), i.e. the assumption is that all processes which take place within the medium are "barotropic".

Liakhov [2] in describing the processes taking place in water saturated soils proposed an equation of state which gives satisfactory agreement with experimental results.

The equation is applicable when there is a high degree of water saturation in the soil, when the gas content is very low and the mineral particles do not form a sufficiently stiff framework, so that the three basic components of the soil, the mineral particles, water and air, are compressed under hydrostatic pressure without any of the significant irreversible increases in density of the medium which are normally associated with the break-up of the framework structure. In this case the equation of state of the mixture of these three components can be obtained, if we know the separate equations of state of the three components and their respective concentrations in the mixture. In his work [2] Liakhov assumed the equation of state of each of the components to be of the type $p = f(\rho)$, i.e. the compression and expansion processes were "barotropic". In the case of mineral particles and water, this is a reasonably good approximation, but in the case of air an isentropic relation between p and ρ is taken, which, is in fact, acceptable in view of the conditions of compression of the air bubbles trapped in the soil.

If now we direct our attention to visco-plastic fluids (clays, other solutions, pastes, etc.) we notice that Liakhov's equations of state are even more accurately fulfilled than in a water saturated soil in which the mineral particles do, after all, form some sort of framework or structure, whereas in our case there is no such structure. We can therefore come to the conclusion that to close the system of equations for a visco-plastic compressible fluid we can use Liakhov's equations of state, i.e. we can relate p and ρ thus:

$$\frac{\rho_0}{\rho} = \alpha_1 \left(\frac{p}{p_0}\right)^{-\varkappa_1} + \alpha_2 \left[\frac{2(p-p_0)k}{\rho_2 c_2^2} + 1\right]^{-\varkappa_2} + \alpha_3 \left[\frac{k_3(p-p_0)}{\rho_3 c_3^2} + 1\right]^{-\varkappa_3} \tag{3}$$
$$\left(\varkappa_1 = \frac{1}{k_1}, \quad \varkappa_2 = \frac{1}{k_2}, \quad \varkappa_3 = \frac{1}{k_3}\right)$$

with the conditions that the equations of state of the components are as follows:

$$\begin{aligned} p &= p_0\left(\frac{\rho}{\rho_1}\right)^{k_1} && \text{(gas)} \\ p &= p_0 + \frac{\rho_2 c_2^2}{k_2}\left[\left(\frac{\rho}{\rho_2}\right)^{k_2} - 1\right] && \text{(liquid)} \\ p &= p_0 + \frac{\rho_3 c_3^2}{k_3}\left[\left(\frac{\rho}{\rho_3}\right)^{k_3} - 1\right] && \text{(minerals)} \end{aligned} \tag{4}$$

and α_1, α_2, α_3, ρ_1, ρ_2, ρ_3 are concentrations and densities respectively for $p = p_0$, i.e. $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\rho_0 = \alpha_1\rho_1 + \alpha_2\rho_2 + \alpha_3\rho_3$ where ρ_0 is the mean density for $p = p_0$, and c_1, c_2, c_3 are velocities of sound in pure components when $p = p_0$.

It should be noted that the form of the equations of state need not be that of (4). With these relations in other forms, the basic expression (3) corresponding thereto will change likewise.

This system of equations enables us to make a study of various dynamical phenomena in visco-plastic media. In particular, in the propagation of powerful disturbances (for instance, those due to explosions, etc.), one can neglect the effect of the parameters and of τ_s so that the characteristic features of shock wave propagation as expressed in (2) will also be valid for visco-plastic media.

A problem of practical interest is that of acoustic propagation within a stream of visco-plastic fluid flowing through a circular pipe. This problem can also be studied with the help of the system of equations described.

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Translated by V.H.B.